MATHEMATICS

REGULAR CONVERGENCE SPACES *)

BY

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It has been shown that regularity of topological spaces is equivalent to a partial converse of the iterated limit theorem, and this converse has been proposed as a definition for regularity of convergence structures [1, 2]. An alternate notion of regularity is this: If $\mathscr{F} \to x$ then $\overline{\mathscr{F}} \to x$, where $\overline{\mathscr{F}}$ is the filter with base of closures of sets in $\mathscr{F}$, with respect to the given convergence structure. FISCHER has shown (Theorem 8, [2]) that this last property implies regularity in the sense mentioned. We use the definitions in Fischer's paper.

Theorem 1: In any regular convergence structure, $\mathscr{F} \to x$ implies that $\overline{\mathscr{F}} \to x$.

Proof: Let (X, τ) be regular, and let $\mathscr{F} \to x$. For each $F \in \mathscr{F}$, let S_F denote the collection of all convergent filters containing the set F. Thus the collection of limits of such filters is $\overline{F}$. Let $\mathscr{S}$ be the filter with base of sets S_F. Clearly these sets are a filter base, as the filters associated with the intersection of two sets F_1 and F_2 is contained in the intersection of S_{F_1} and S_{F_2}. Now we examine $\varkappa(\mathscr{S})$. It is generated by sets of the form

$$A_{F,\varphi} = \cup\, \varphi(\mathscr{D}) : \mathscr{D} \in S_F;\ \ \varphi(\mathscr{D}) \in \mathscr{D}$$

and for each $F \in \mathscr{F}$, the function $\varphi(\mathscr{D}) = F \in \mathscr{D}$ yields just the set F. Hence $\mathscr{F} \leqslant \varkappa(\mathscr{S})$, and as $\mathscr{F}$ converges, we see that $\varkappa(\mathscr{S})$ also converges.

Now we can apply the converse of the iterated limit theorem to conclude that the associated filter of limits of the filters in $\mathscr{S}$, converges to x. But the filter of limits has the sets $\overline{F}$ as a base, so we have $\overline{\mathscr{F}} \to x$.

q.e.d.

Thus we have shown that the two notions of regularity are equivalent, for convergence structures, in view of Fischer's results. For the result above, we do not even require the ideal property; merely that refinements of convergent filters converge.

The fact that a partially ordered set is a regular convergence structure (as a substructure of its completion, with order convergence) was shown in [3]. The present theorem equates the alternative formulation of regu-

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larity for that system. $\alpha\mathscr{F}$ denotes the adherence of the filter $\mathscr{F}$. We give a simple

Proposition: For topological spaces, if for $\mathscr{F} \in \mathbb{F}(X)$, $\alpha\mathscr{F}=\{x\}$ if and only if $\mathscr{F} \to x$ then the space is regular and separated.

Proof: If $\mathscr{F} \to x$ then $\alpha\mathscr{F}=\cap\,\{\bar{F} : F \in \mathscr{F}\}=\{x\}$; $\bar{\bar{F}}=\bar{F}$, so

$$\{x\}=\cap\,\{\bar{S} : S \supset F,\ F \in \mathscr{F}\}=\alpha\mathscr{F} \text{ so } \mathscr{F} \to x.$$

q.e.d.

The converse holds in compact spaces.

Condition A:

$$\alpha\mathscr{F}=\cap\,\{\bar{F} : F \in \mathscr{F}\}.$$

In general for a convergence structure, $\alpha\mathscr{F}$ is this set of limits of refinements of $\mathscr{F}$; we will restrict attention to those spaces satisfying the condition above.

We give a form of DEMARR's result on o-spaces [4], but for some convergence spaces.

Theorem 2: Let (X, τ) be a separated regular convergence structure in which condition A holds. Let (X, τ) be principal. Then (X, τ) is a subspace of a complete lattice with order convergence.

For the proof we use the notation of DeMarr's note.

Proof: Let $\mathfrak{F}=\{\bar{F}^{\tau} : F \subset X\}$ and form Ω_0 and $\bar{\Omega}$ as in [4], using this different class $\mathfrak{F}$. $\bar{\Omega}$ is again a complete lattice. We write $\psi : X \to \Omega_0$ with $\psi(x)=(0, x)$. Let $\{x_n\} \to {}^{\tau}x$; we show that $\{\psi(x_n)\} \to {}^{\circ}\, \psi(x)$. For write $\mathscr{F}$ for the elementary filter of the net $\{x_n\}$. Then $\mathscr{F} \to {}^{\tau}x$, so $\{x\}=\cap\,\bar{F} : F \in \mathscr{F}$, by condition A. Thus the net $\{(0, x_n)\}$ is eventually bounded above by $(1, \bar{F})$, for each $F \in \mathscr{F}$. $x \in \bar{F}$, for each $F \in \mathscr{F}$, so $(0, x) \leqslant (1, \bar{F})$. As $\cap\,\bar{F}=\{x\}$, we have $(0, x)=\inf\,\{(1, \bar{F}) : F \in \mathscr{F}\}$. Similarly $(0, x)=\sup\,\{(-1, \bar{F}) : F \in \mathscr{F}\}$. Thus we see that $\{(0, x)\} \to {}^{\circ}\,(0, x)$.

Conversely, if $\{x_n\} \not\to {}^{\tau}x$, then there exists an $N \in \mathfrak{N}(x)$ such that $\{x_n\}$ is frequently in $\mathscr{C}N$; let $E=\overline{\mathscr{C}N}$; $x \notin E$ as x has a neighborhood which does not meet $\mathscr{C}N$, and hence by DeMarr's calculations, $\liminf \psi(x_n) \leqslant (1, E)$, and $(0, x)$ not $\leqslant (1, E)$, so $x_n \not\to^{\circ} x$.

q.e.d.

Observe that τ-convergence implies o-convergence without any assumption that the space be principal. We have given the usual proof of the other implication. An extremely easy proof of that can be given for those topological spaces covered by our theorem. For if $\mathscr{F} \to^{\circ} x$, we observe that for each $F \in \mathscr{F}$, $(1, \bar{F})=\sup\,\{(0, y) : y \in F\}$ in the given po-set. Hence, as we have $(\cap\,\bar{F} : F \in \mathscr{F})$ closed, either this is $\{x\}$ or else the pair $(1, (\cap\,\bar{F} : F \in \mathscr{F}))$ is a lower bound to the sets $(1, \bar{F})$ and is greater than $(0, x)$. This completes the proof in the topological case. Of course it is convenient to have both notions of regularity since our earlier proof of regularity of convergence lattices used the net formulation of regularity.

It is quite easy to give examples of lattices where the convergent filters at a point do not form an ideal, much less a principal ideal—hence we certainly have not characterized the convergence structures of lattices. On the other hand every complete lattice is compact. If we take a closed principal subspace, it is again compact. In compact separated principal convergence structures condition A holds. Thus every closed principal subspace of a complete lattice with order convergence satisfies the hypotheses of Theorem 2.

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